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DDF construction and D-brane boundary states in pure spinor formalism

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Abstract: Open string boundary conditions for non-BPS D-branes in type II string theories discussed in hep-th/0505157 give rise to two sectors with integer (R sector) and half-integer (NS sector) modes for the combined fermionic matter and bosonic ghost variables in pure spinor formalism. Exploiting the manifest supersymmetry of the formalism we explicitly construct the DDF (Del Giudice, Di Vecchia, Fubini) states in both the sectors which are in one-to-one correspondence with the states in light-cone Green-Schwarz formalism. We also give a proof of validity of this construction. A similar construction in the closed string sector enables us to define a physical Hilbert space in pure spinor formalism which is used to project the covariant boundary states of both the BPS and non-BPS instantonic D-branes. These projected boundary states take exactly the same form as those found in light-cone Green-Schwarz formalism and are suitable for computing the cylinder diagram with manifest open-closed duality.

Keywords: D-branes, Superstrings and Heterotic Strings.

Contents

1. Introduction and summary

It has been a long standing problem of fundamental interest to quantise superstring theory with all the space-time symmetries manifest until Berkovits' proposal of pure spinor formalism [1] was put forward. The formalism comes with a bag of tools which includes a conformally gauge fixed action with total central charge zero, a BRST operator, physical state condition and rules for computing scattering amplitudes. It is remarkable that everything fits together to give consistent results like spectrum of physical states and super-Poincaré covariant results for the scattering amplitudes in flat space [2−4].[1] Although the space-time symmetry is very much emphasised, it is not very clear what role the world-sheet conformal symmetry has to play in this case, something which is very transparent in the Neveu-Schwarz-Ramond (NSR) formalism. A particular example is to understand

[1] See also [5].

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D-branes[2] from both the open and closed string point of view. In NSR formalism these two views are bridged by the underlying world-sheet picture where the modular transformation relates them. This picture is not, in general, apparent if one is armed only with a BRST setup like in pure spinor formalism. Let us consider the simplest diagram, namely the cylinder, which computes the force between two D-branes. Following will be the generic procedure to compute this in a BRST setup: start out with the quadratic space-time action involving the BRST operator with a linearised gauge invariance. Obtain the propagator by inverting the kinetic term with a valid gauge choice. Then compute the relevant Feynman diagram where two external sources are connected by a single line (see, for example, [8]). If we know the correct strength for all the sources then this computation is completely well-defined, the only problem being there are infinite number of fields to be taken into account. In this computation we do not use any CFT techniques as there is no world-sheet interpretation and therefore the open-closed duality is not manifest. In NSR formalism this interpretation results from a simple gauge choice [9] which we call Siegel gauge.[3] It is in this gauge the closed string propagator in Schwinger parametrisation has the interpretation of world-sheet time evolution. It is not clear what would be the corresponding gauge choice in pure spinor formalism.

In more technical terms the problem can be described in the following way. The cylinder diagram is computed in the closed string channel by first constructing the boundary state in Siegel gauge and then computing an inner product where the world-sheet time evolution operator is sandwiched between two boundary states. The result can then be interpreted in the open string channel by performing a modular transformation. There are two basic ingredients in this computation:

1. A suitable boundary state that provides the correct source terms for all the relevant closed string states.
2. The correct choice of degrees of freedom that should be allowed to propagate along the cylinder (which is implemented by the Siegel gauge).

In pure spinor formalism the first problem can be solved without much trouble. Although there is a pure spinor constraint on the bosonic ghost sector which makes the construction of boundary states troublesome [10], it has been suggested in [11] that constructing the boundary states in the relaxed CFT where there is no constraint also does the job. Writing down the boundary conditions and boundary states in the free CFT simply bypasses the technical difficulty of incorporating the pure spinor constraint yet producing the correct results for the source terms once the rules for such computations are set up properly [11]. The main point of doing this is the fact that it is not beneficial to go through the technical difficulty of imposing the pure spinor constraint as this, by itself, does not solve the second problem. To solve the second problem one also needs to throw away the gauge degrees of freedom. This claim has been explicitly demonstrated in [11] through a computation of the long range force between two D-branes. If it requires us to gauge fix the space-time

[2] D-branes have been studied from various points of view in pure spinor formalism in [6, 10].

[3] Its bsonic counterpart is called Siegel gauge in string field theory analysis.

theory at every mass level separately then that would be an uncontrollable job to do. A full string theoretic treatment of allowing only the correct degrees of freedom to propagate seems to be a subtle issue.

Certainly the above subtlety is encountered when one tries to do the computation with full covariance under SO(9,1). Here we shall show that the cylinder diagram can indeed be computed in pure spinor formalism more easily by preserving covariance only under the transverse SO(8) part. The approach will be as follows: since the computation is well understood in the light-cone Green-Schwarz (LCGS) formalism, it will suffice us to construct the LCGS boundary states [12 – 15] explicitly in pure spinor formalism. In other words if the LCGS Hilbert space can be constructed explicitly in pure spinor formalism, then any covariant pure spinor boundary state could be projected onto that Hilbert space. The projected boundary states could then be evolved by the world-sheet time evolution. By exploiting the manifest space-time supersymmetry of pure spinor formalism we construct the LCGS Hilbert space, which will be denoted $\mathcal{H}_{\rm DDF}$, by going through the analogue of well-known DDF (Del Giudice, Di Vecchia, Fubini) construction [16].

For open strings on a BPS D-brane this construction is done by first using ghost number one, dimension zero unintegrated massless vertex operators to construct certain massless physical states in the vector and conjugate spinor representations of SO(8) with special kinematical condition that the light-cone component q^+ of the momentum is non-zero and fixed. Then we use the ghost number zero, dimension one integrated massless vertex operators to construct the DDF operators which are the analogues of the LCGS oscillators. These operators commute/anticommute with the BRST operator so that while acting on physical states they produce other physical states. The DDF operators constructed this way have nontrivial expansions in terms of the fermionic matter variable θ. Therefore, although the leading contribution to the DDF commutation relations do match with that of the LCGS oscillators [18], there are terms higher order in θ. We define the physical Hilbert space $\mathcal{H}_{\rm DDF}$ to be spanned by all the states which are obtained by applying creation modes on the massless physical states constructed in the first step. The ghost number two conjugate states are similarly constructed by applying DDF operators on certain massless states that form the BRST cohomology at ghost number 2. These states are chosen so that they are conjugate to the ghost number one massless DDF states. We prove that the DDF states constructed this way form an orthonormal basis in $\mathcal{H}_{\rm DDF}$. The orthogonality of the DDF states establishes the fact that all the higher order θ-terms drop off when the DDF commutators are restricted in $\mathcal{H}_{\rm DDF}$, so that the commutation relations exactly match with those of LCGS oscillators. This implies that though the DDF operators constructed here have complicated θ-expansions they behave as simply as LCGS oscillators in $\mathcal{H}_{\rm DDF}$.

For a non-BPS D-brane, we have argued, using the boundary conditions suggested in [11], that there are two sectors of open strings - R and NS sectors. All the world-sheet fields that are space-time fermions (i.e. fermionic matter and bosonic ghost) satisfy periodic and anti-periodic boundary conditions on the doubled surface in R and NS sectors respectively. DDF construction for the R sector goes through as described in the previous paragraph. For the NS sector the bosonic DDF operators are constructed in the same way. But the fermionic ones are constructed in a slightly different way so that they have half-integer modes instead of integer modes. This sector has a unique ground state which is included in the BRST cohomology. We identify this with the open string tachyon. This way the DDF states in the NS sector gives an explicit construction of the corresponding open string spectrum found in LCGS formalism [19, 15].

Doing the similar construction on the closed string sector we define the physical Hilbert space $\mathcal{H}_{\mathrm{DDF}}$ on the closed string side. Given the covariant boundary states constructed in the free CFT, as in [11], projection of the corresponding actual pure spinor boundary states onto $\mathcal{H}_{\mathrm{DDF}}$ can be constructed unambiguously. Due to the special kinematical condition of the DDF states that all of them have a fixed nonzero q^+, only instantonic D-brane boundary states can be projected this way to get the physical components. To practically derive a projected boundary state we proceed as follows. Since a projected boundary state is supposed to be expanded in terms of the DDF states, it should be possible to get this as a solution to the gluing conditions satisfied by the DDF operators. Starting from the boundary conditions written in the open string channel we derive the DDF gluing conditions and show that they are given by the same equations satisfied by the LCGS oscillators as discussed in [12, 14, 15]. Therefore the projected boundary states in pure spinor formalism are obtained from the corresponding boundary states in LCGS formalism simply by interpreting the LCGS oscillators as the DDF operators constructed here. These boundary states can then be evolved by the world-sheet time evolution in pure spinor formalism to give the correct result for the cylinder. In all our discussion we shall consider type IIB string theory for definiteness and work in the $\alpha' = 2$ unit. Generalisation to type IIA is straightforward.

The rest of the paper is organised as follows: section 2 reviews the non-BPS boundary conditions as suggested in [11] and analyses the open string spectrum. Section 3 discusses the DDF construction for open strings on both BPS and non-BPS D-branes of type II string theories. This also includes a proof of validity of the construction. Section 4 defines the projected boundary states and derives the DDF gluing conditions. The line of argument to compute the cylinder diagram has been given in section 5. We conclude with a few unresolved questions in section 6. Several appendices contain necessary technical details.

2. Boundary conditions and spectrum of open strings on non-BPS D-branes

2.1 Review of boundary conditions

SO(8) covariant open string boundary conditions for non-BPS D-branes in LCGS formalism were obtained in [15]. Generalising this work to any manifestly supersymmetric formalism, similar boundary conditions were suggested in pure spinor formalism in [11]. Specialising to type IIB string theory, these boundary conditions take the following form for the combined fermionic matter and bosonic ghost sector in the unconstrained CFT,

$$\left.\begin{aligned} U^\alpha(z)U^{\beta T}(w) &= \mathcal{M}^{\alpha\beta}_{\gamma\delta}\tilde{U}^\gamma(\bar{z})\tilde{U}^{\delta T}(\bar{w})\,,\\ U^\alpha(z)V^T_\beta(w) &= \mathcal{M}^{\alpha}{}_{\beta\gamma}{}^{\delta}\tilde{U}^\gamma(\bar{z})\tilde{V}^T_\delta(\bar{w})\,,\\ V_\alpha(z)V^T_\beta(w) &= \mathcal{M}^{\gamma\delta}_{\alpha\beta}\tilde{V}_\gamma(\bar{z})\tilde{V}^T_\delta(\bar{w})\,,\end{aligned}\right\}\ \text{at } z=\bar{z}, w=\bar{w}\,, \tag{2.1}$$

where we have introduced the column vectors,

$$U^\alpha(z) = \begin{pmatrix} \lambda^\alpha(z) \\ \theta^\alpha(z) \end{pmatrix} , \quad V_\alpha(z) = \begin{pmatrix} w_\alpha(z) \\ p_\alpha(z) \end{pmatrix} , \tag{2.2}$$

and similarly for the right moving sector. The coupling matrices are given by,

$$\begin{aligned}
\mathcal{M}^{\alpha\beta}_{\gamma\delta} &= -\Bigg[\frac{1}{16}\gamma^{\alpha\beta}_{\mu}(M^V)^{\mu}{}_{\nu}\bar{\gamma}^{\nu}_{\gamma\delta} + \frac{1}{16\times 3!}\gamma^{\alpha\beta}_{\mu_1\cdots\mu_3}(M^V)^{\mu_1}{}_{\nu_1}\cdots(M^V)^{\mu_3}{}_{\nu_3}\bar{\gamma}^{\nu_1\cdots\nu_3}_{\gamma\delta} \\
&\quad + \frac{1}{16\times 5!}\sum_{\mu_1,\ldots,\mu_5\in\mathcal{K}^{(5)}}\gamma^{\alpha\beta}_{\mu_1\cdots\mu_5}(M^V)^{\mu_1}{}_{\nu_1}\cdots(M^V)^{\mu_5}{}_{\nu_5}\bar{\gamma}^{\nu_1\cdots\nu_5}_{\gamma\delta}\Bigg] , \\
\mathcal{M}^{\alpha}{}_{\beta\gamma}{}^{\delta} &= \frac{1}{16}\delta^{\alpha}{}_{\beta}\delta_{\gamma}{}^{\delta} + \frac{1}{16\times 2!}\gamma_{\mu_1\mu_2}{}^{\alpha}{}_{\beta}(M^V)^{\mu_1}{}_{\nu_1}(M^V)^{\mu_2}{}_{\nu_2}\bar{\gamma}^{\nu_1\nu_2}{}_{\gamma}{}^{\delta} \\
&\quad + \frac{1}{16\times 4!}\gamma_{\mu_1\cdots\mu_4}{}^{\alpha}{}_{\beta}(M^V)^{\mu_1}{}_{\nu_1}\cdots(M^V)^{\mu_4}{}_{\nu_4}\bar{\gamma}^{\nu_1\cdots\nu_4}{}_{\gamma}{}^{\delta} .
\end{aligned} \tag{2.3}$$

Our gamma matrix conventions can be found in appendix A. The summation convention for the repeated indices has been followed for all the terms in the above two equations except for the last term of the first equation. The sum over the five vector indices $\mu_1\cdots\mu_5$ has been restricted to a set $\mathcal{K}^{(5)}$ which is defined as follows. We divide the set of all possible sets of five indices $\{\{\mu_1,\ldots,\mu_5\}|\mu_i = 0,\ldots,9\}$ into two subsets of equal order, namely $\mathcal{K}^{(5)}$ and $\mathcal{K}^{(5)}_D$ such that for every element $\{\mu_1,\ldots,\mu_5\} \in \mathcal{K}^{(5)}$ there exists a dual element $\{\mu_1,\ldots,\mu_5\}_D = \{\nu_1,\ldots,\nu_5\} \in \mathcal{K}^{(5)}_D$ such that, $\epsilon^{\mu_1\cdots\mu_5\nu_1\cdots\nu_5} \neq 0$. The supersymmetry currents, being odd in the world-sheet fields belonging to the combined fermionic matter and bosonic ghosts, do not satisfy a linear boundary condition. But the above boundary conditions do lead to BRST invariance.

Following the method of [15] we can now introduce the holomorphic fields $\mathcal{U}^\alpha(z)$ and $\mathcal{V}_\beta(z)$ on the doubled surface through the following expressions,

$$\mathcal{U}^\alpha(u)\cdots\mathcal{U}^{\beta T}(v) = \begin{cases} \mathcal{U}^\alpha(z)\cdots\mathcal{U}^{\beta T}(w)|_{z=u,w=v} , & \Im u, \Im v \geq 0 , \\ \mathcal{M}^{\alpha\beta}_{\gamma\delta}\tilde{\mathcal{U}}^\gamma(\bar{z})\cdots\tilde{\mathcal{U}}^{\delta T}(\bar{w})|_{\bar{z}=u,\bar{w}=v} , & \Im u, \Im v \leq 0 , \end{cases} \tag{2.4}$$

$$\mathcal{U}^\alpha(u)\cdots\mathcal{V}^T_\beta(v) = \begin{cases} \mathcal{U}^\alpha(z)\cdots\mathcal{V}^T_\beta(w)|_{z=u,w=v} , & \Im u, \Im v \geq 0 , \\ \mathcal{M}^{\alpha}{}_{\beta\gamma}{}^{\delta}\tilde{\mathcal{U}}^\gamma(\bar{z})\cdots\tilde{\mathcal{V}}^T_\delta(\bar{w})|_{\bar{z}=u,\bar{w}=v} , & \Im u, \Im v \leq 0 , \end{cases} \tag{2.5}$$

$$\mathcal{V}_\alpha(u)\cdots\mathcal{V}^T_\beta(v) = \begin{cases} \mathcal{V}_\alpha(z)\cdots\mathcal{V}^T_\beta(w)|_{z=u,w=v} , & \Im u, \Im v \geq 0 , \\ \mathcal{M}^{\gamma\delta}_{\alpha\beta}\tilde{\mathcal{V}}_\gamma(\bar{z})\cdots\tilde{\mathcal{V}}^T_\delta(\bar{w})|_{\bar{z}=u,\bar{w}=v} , & \Im u, \Im v \leq 0 . \end{cases} \tag{2.6}$$

The dots imply that the relations are considered to be true even when other operators appear in between in a correlation function. As argued in [15], an immediate consequence of the above definitions is,

$$\mathcal{U}^\alpha(\tau, 2\pi) = \pm\mathcal{U}^\alpha(\tau, 0) , \quad \mathcal{V}_\alpha(\tau, 2\pi) = \pm\mathcal{V}_\alpha(\tau, 0) , \tag{2.7}$$

so that there are two sectors of open strings, namely the R (periodic) and the NS (anti-periodic) sectors. Below we shall discuss the spectrum of these open strings.

2.2 Open string spectrum

The periodic sector can be analysed in the usual way [1] and therefore will give rise to an open string spectrum which is same as that on a BPS D-brane of same dimensionality in type IIA theory. Therefore the anti-periodic sector will be our main topic of discussion here. In this sector all the relevant fields have half-integer modes: $\mathcal{U}_r^\alpha$ and $\mathcal{V}_{\alpha,r}$, with $r \in \mathbf{Z} + 1/2$. Therefore, in absence of any zero modes, there is a unique ground state $|\sigma\rangle$ in the combined fermionic matter and bosonic ghost sector, defined in the following way,

$$\mathcal{U}_r^\alpha|\sigma\rangle = 0\,, \quad \mathcal{V}_{\alpha,r}|\sigma\rangle = 0\,, \qquad \forall r \geq 1/2\ . \tag{2.8}$$

We may define the ghost number for this state to be one. Excited states in the theory are obtained by applying the negative modes of the oscillators on $|\sigma\rangle$ and by imposing pure spinor constraint. Physical states are the ghost number one states in BRST cohomology, where the BRST operator (see eq. (A.9)) takes the following form in terms of various modes in $\alpha' = 2$ unit,

$$Q_B = \sum_r (\lambda_{-r} p_r) + \frac{i}{2}\sum_{r,s} (\lambda_{-r}\bar\gamma_\mu\theta_{-s})\,\alpha^\mu_{r+s} - \frac{1}{8}\sum_{r,s,t}(r+s+t)\,(\lambda_r\bar\gamma^\mu\theta_s)\,(\theta_t\bar\gamma_\mu\theta_{-r-s-t})\ , \tag{2.9}$$

where $\alpha_n^\mu = \oint \frac{dz}{2\pi} z^n \partial X^\mu(z)$. Mass of the state is determined by the fact that the state is annihilated by the Virasoro zero mode L_0 given in eq. (A.12). It is easy to check that the ground state $|\sigma, k\rangle$, with momentum k, is an allowed state in the BRST cohomology.

$$Q_B|\sigma,k\rangle = 0\,,\ (\text{for any } k)\,, \qquad L_0|\sigma,k\rangle = 0\,, \Rightarrow M^2 = -\frac{1}{4}\ . \tag{2.10}$$

This is the open string tachyon. There is a space-time fermion at the massless level. The chirality is same as that of the massless fermion in the R-sector.[4]

$$\begin{aligned} |\xi(k),k\rangle &\equiv \xi^\alpha(k) p_{\alpha,-1/2}|\sigma,k\rangle\,,\\ Q_B|\xi(k),k\rangle = 0 &\Rightarrow k^\mu\left(\bar\gamma_\mu\xi(k)\right)_\alpha = 0\,,\\ L_0|\xi(k),k\rangle = 0 &\Rightarrow k^2 = 0\ . \end{aligned} \tag{2.11}$$

One can further proceed in the similar way. But to count all the states in the BRST cohomology one may proceed to follow the argument of Berkovits given in [2]. We shall not repeat the details here. But one can argue that the analysis goes through for the NS sector simply by considering all the SO(8) vectors to be periodic and all the SO(8) spinors to be anti-periodic on the world-sheet. This includes all the fields appearing in the analysis of [2] including the infinite number of ghosts for ghosts. For example the vector field in eqs.(4.3) of [2] will have the same mode expansion, but the spinor field, in the present case, will be expanded in terms of half-integer modes. Having constructed all

[4]The correct way to determine the chirality is to first realise that these massless fermions are the goldstinos corresponding to the spontaneously broken space-time supersymmetries [19]. This tells us that they should have opposite chirality in type IIA theory which, in turn, determines the chirality of fermionic matter and bosonic ghost fields that need to be used for the NS sector.

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the transverse creation modes, all the elements in the BRST cohomology can be obtained simply by applying those operators freely with a factor of[5] $e^{-ik^-X^+}$, with $k^- = \frac{N}{k^+}$ on the unique ground state $|\sigma, k\rangle$ with $k^+ \neq 0$ (fixed), $k^- = 0$ and $\vec{k}^2 = 1$. Here N is the total level of all the creation operators. Notice that $|\sigma, k\rangle$ is the NS-sector analogue of the state in eq. (4.8) of [2] and it does not have the c-ghost dependent additive part. This is because $|\sigma, k\rangle$ is annihilated by Q_B even without the pure spinor constraint as can be checked by using eqs. (2.8) and (2.9).

Although the arguments given in [2] applied to the NS sector establish the fact that the pure spinor BRST cohomology is isomorphic to the LCGS Hilbert space [15], we do not have an explicit construction of this Hilbert space in terms of the pure spinor degrees of freedom. In the next section we shall achieve this by performing the analogue of DDF construction in pure spinor formalism without introducing the infinite number of ghosts for ghosts.

3. The DDF construction

Here we shall first construct the DDF states for the periodic sector. The same for the anti-periodic sector, which will be discussed next, can be constructed more easily.

3.1 Periodic sector

The construction is done using the massless vertex operators both in unintegrated and integrated forms as discussed in appendix B.[6] Unlike in the NSR formalism [18], we exploit the manifest supersymmetry in the present case to build the whole construction. All the DDF states will be ghost number one physical states with manifest SO(8) covariance and will be in one-to-one correspondence with the states in LCGS formalism. The construction goes through the following two steps:

1. Construct the SO(8) covariant massless DDF states by using the BRST and supersymmetry properties of the gluon and gluino vertex operators in the unintegrated form. These operators have dimension zero and ghost number one. The conjugate states, with respect to which the massless DDF states form an orthonormal basis, are constructed using the unintegrated vertex operators of ghost number two states in the BRST cohomology.

2. Construct the BRST invariant DDF operators that are in one-to-one correspondence with the bosonic and fermionic oscillators in LCGS formalism by using the dimension one, ghost number zero gluon and gluino vertex operators in the integrated form. One obtains the physical Hilbert space $\mathcal{H}_{\rm DDF}$ by applying the creation DDF modes on the massless states constructed in the first step. Similarly the conjugate states are obtained by applying the DDF operators on the massless conjugate states.

[5] In our notation the mass-shell condition for the anti-periodic sector reads: $M^2 = -\frac{k^2}{4} = \frac{1}{4}(2k^+k^- - \vec{k}^2) = \frac{1}{2}(N - \frac{1}{2})$.

[6] See [17] for a detailed discussion on closed string vertex operators.

In order for the above construction to go through one needs to show that the DDF operators do have the same algebra of the corresponding LCGS oscillators. We shall argue that this condition is indeed satisfied within $\mathcal{H}_{\rm DDF}$ by proving that the DDF states constructed above form an orthogonal basis in $\mathcal{H}_{\rm DDF}$.

Let us now proceed to perform the first step. We shall consider the massless unintegrated vertex operator in (B.1) with the gauge choice,

$$a^+(k) = 0\,, \tag{3.1}$$

and the choice of momentum: $k^+(\neq 0)$ and k^I left arbitrary and $k^- = \frac{\vec{k}^2}{2k^+}$. According to the on-shell conditions in (B.3), this implies,

$$a^-(k) = \frac{1}{k^+}k^I a^I(k)\,, \qquad \xi_L(k) = -\frac{1}{\sqrt{2}k^+}k^I\sigma^I\xi_R(k)\ . \tag{3.2}$$

To simplify notations we have depicted the above quantities as functions of k, whereas they are actually functions of only k^+ and $\vec{k}$. We shall follow the same notation below as well. Using the above conditions in eq. (B.1) one gets,

$$\begin{aligned} u(k,z) &= a^I(k)v^I(k,z) + \xi_R^{\dot a}(k)s_R^{\dot a}(k,z)\,,\\ v^I(k,z) &= b^I(k,z) - \frac{k^I}{k^+}b^+(k,z)\,,\\ s^{\dot a}(k,z) &= f_R^{\dot a}(k,z) - \frac{k^I}{\sqrt{2}k^+}\sigma^I_{a\dot a}f_L^a(k,z)\,, \end{aligned} \tag{3.3}$$

where $a^I(k)$ and $\xi_R^{\dot a}(k)$ are the independent components. Therefore the states,

$$|I,k\rangle \equiv v^I(k,0)|0\rangle\,, \qquad |\dot a,k\rangle \equiv \sqrt{\frac{k^+}{i\sqrt{2}}}\ s^{\dot a}(k,0)|0\rangle\,, \tag{3.4}$$

are the physical ground states in 8_v and 8_c of SO(8). The particular normalisation chosen will be explained later. Let us now discuss the supersymmetry transformations of these states. Using eqs. (B.5) one finds that in terms of the SO(8) notations these are given by (up to BRST exact terms),

$$\begin{aligned} Q_L^a|I,k\rangle &= \sqrt{\frac{k^+}{i\sqrt{2}}}\sigma^I_{a\dot a}|\dot a,k\rangle\,,\\ Q_L^a|\dot a,k\rangle &= \sqrt{\frac{k^+}{i\sqrt{2}}}\sigma^I_{a\dot a}|I,k\rangle\,,\\ Q_R^{\dot a}|I,k\rangle &= \frac{1}{\sqrt{i2\sqrt{2}k^+}}k^J(\delta^{JI}\delta_{\dot a\dot b} + \bar\sigma^{JI}_{\dot a\dot b})|\dot b,k\rangle\,,\\ Q_R^{\dot a}|\dot b,k\rangle &= \frac{1}{\sqrt{i2\sqrt{2}k^+}}k^I(\delta^{IJ}\delta_{\dot a\dot b} + \bar\sigma^{IJ}_{\dot a\dot b})|J,k\rangle\ . \end{aligned} \tag{3.5}$$

To show the last equality one uses the fact that $k_\mu b^\mu(k,z)$ is BRST exact which is responsible for the gauge invariance in (B.3). The above equations are precisely the supersymmetry transformations of the massless states in LCGS formalism [18]. One can also define a set of conjugate states $\langle I, k|$ and $\langle \dot{a}, k|$ with the following inner products:

$$\langle I,k|J,l\rangle = \delta^{IJ}\delta(k^+ + l^+)\delta^8(\vec{k}+\vec{l}) , \qquad \langle \dot{a}, k^+|\dot{b}, l^+\rangle = \delta^{\dot{a}\dot{b}}\delta(k^+ + l^+)\delta^8(\vec{k}+\vec{l}) \ . \quad (3.6)$$

These states can be explicitly constructed in terms of ghost number two zero-mode operators once the θ-expansions of the states $|I,k\rangle$ and $|\dot{a},k\rangle$ are known. These states are also annihilated by Q_B and form the BRST cohomology at ghost number two.

We shall now proceed to the second step where the DDF operators will be constructed using the vertex operators $\mathcal{B}^\mu(k,z)$ and $\mathcal{F}_\alpha(k,z)$ (see appendix B). Let us first consider the gluon vertex operator $\mathcal{B}^I(k^-,z)$ along the light-cone transverse direction $\mu = I$ with momentum: $k^- \neq 0, k^+ = k^I = 0$. Using eqs. (B.7) one shows that only the left moving gluino is generated under the space-time supersymmetry transformations of this operator.

$$\left[Q_L^a, \mathcal{B}^I(k^-,z)\right] = 0, \qquad \left[Q_R^{\dot{a}}, \mathcal{B}^I(k^-,z)\right] = -\frac{i}{\sqrt{2}}k^-\sigma^I_{a\dot{a}}\mathcal{F}^a_L(k^-,z) , \quad (3.7)$$

The supersymmetry transformations of $\mathcal{F}_L(k^-,z)$ are given by (up to total derivative terms),

$$\left\{Q_L^a, \mathcal{F}^b_L(k^-,z)\right\} = \sqrt{2}\delta^{ab}\mathcal{B}^+(k^-,z) , \qquad \left\{Q_R^{\dot{a}}, \mathcal{F}^a_L(k^-,z)\right\} = \sigma^I_{a\dot{a}}\mathcal{B}^I(k^-,z) \ . \quad (3.8)$$

Notice that $\mathcal{B}^+(0,z) = \partial X^+(z)$ (eq. (B.9)) and for non-zero k^-, $\mathcal{B}^+(k^-,z)$ should be a total derivative as $k_\mu \mathcal{B}^\mu(k,z)$ is so, which is responsible for the gauge invariance in (B.3). Therefore only the first term in the first equation in (B.8) will survive at the lowest θ-level. Moreover, the higher order θ-terms can not contribute as the operator has to be a dimension one total derivative. Therefore we should have:

$$\mathcal{B}^+(k^-,z) = \frac{i}{k^-}\partial e^{-ik^-X^+}(z) , \qquad k^- \neq 0 \ . \quad (3.9)$$

Next we define the DDF operators,

$$\begin{aligned} A^I_n(k_0) &\equiv \oint \frac{dz}{2\pi}\mathcal{B}^I(k^- = -nk_0, z) , \\ S^a_n(k_0) &\equiv \frac{1}{\sqrt{-i\sqrt{2}\alpha_0^+}}\oint \frac{dz}{2\pi i}\mathcal{F}^a_L(k^- = -nk_0, z) , \end{aligned} \quad (3.10)$$

where k_0 is a real number and α_0^+ follows from the definition given below eq. (2.9). All the DDF states are going to have a fixed value of α_0^+ which is same as that of the states in (3.4). This is simply because none of the above DDF operators excites momentum along this direction. Also we shall argue in appendix C that ∂X^- does not appear in any of the DDF operators (see statement (C.8)), so that α_0^+ appears to be only a c-number in the string of DDF operators in a given DDF state. Being constructed out of dimension one primaries, the DDF operators commute with all the Virasoro generators. In particular, commuting with L_0 implies that the action of the n-th mode changes the level of a state by $-n$. Using eqs. (B.6) one can argue that while acting on any state of momentum q, the operators $A^I_n(1/q^+)$ and $S^a_n(1/q^+)$ are BRST invariant.

$$\big[Q_B, A^I_n(1/q^+)\big] = 0\,, \qquad \big\{Q_B, S^a_n(1/q^+)\big\} = 0\,. \tag{3.11}$$

The supersymmetry transformations of the DDF operators take the following form on any such state,

$$\begin{aligned}
\big[Q^a_L, A^I_n(1/q^+)\big] &= 0\,, & \big[Q^{\dot a}_R, A^I_n(1/q^+)\big] &= -\frac{n}{\sqrt{i\sqrt{2}q^+}} S^a_n(1/q^+)\,, \\
\big\{Q^a_L, S^b_n(1/q^+)\big\} &= \sqrt{-i\sqrt{2}q^+}\delta^{ab}\delta_{n,0}\,, & \big\{Q^{\dot a}_R, S^a_n(1/q^+)\big\} &= \frac{1}{\sqrt{i\sqrt{2}q^+}}\sigma^I_{a\dot a} A^I_n(1/q^+)\,.
\end{aligned} \tag{3.12}$$

Although the above transformations are precisely the ones expected for the LCGS oscillators, the commutation relations among the DDF operators do not quite form the desired algebra because of the terms higher order in θ. Using various OPE's in appendix A one can show:

$$\begin{aligned}
\mathrm{Res}_{z\to w}\mathcal{B}^I(k^-,z)\mathcal{B}^J(p^-,w) &= ik^-\delta^{IJ}\partial X^+(w)e^{-i(k^-+p^-)X^+}(w) + \mathcal{O}(\theta^2)\,, \\
\mathrm{Res}_{z\to w}\mathcal{F}^a_L(k^-,z)\mathcal{F}^b_L(p^-,w) &= \sqrt{2}\delta^{ab}\partial X^+(w)e^{-i(k^-+p^-)X^+}(w) + \mathcal{O}(\theta^2)\,, \\
\mathrm{Res}_{z\to w}\mathcal{B}^I(k^-,z)\mathcal{F}^a_L(p^-,w) &= 0 + \mathcal{O}(\theta)\,,
\end{aligned} \tag{3.13}$$

which imply the following commutation relations,

$$\begin{aligned}
\big[A^I_m(1/q^+), A^J_n(1/q^+)\big] &= m\delta^{IJ}\delta_{m,-n} + \mathcal{O}(\theta^2)\,, \\
\Big\{S^a_m(1/q^+), S^b_n(1/q^+)\Big\} &= \delta^{ab}\delta_{m,-n} + \mathcal{O}(\theta^2)\,, \\
\big[A^I_m(1/q^+), S^a_n(1/q^+)\big] &= 0 + \mathcal{O}(\theta)\,.
\end{aligned} \tag{3.14}$$

We shall argue later that these higher order terms will drop off in the physical Hilbert space that we are going to define. We first define the following excited states,

$$\left.\begin{array}{l} |\{(I_i,n_i)\},\{(a_j,m_j)\},I,q\rangle \\ |\{(I_i,n_i)\},\{(a_j,m_j)\},\dot a,q\rangle \end{array}\right\} \propto \prod_i A^{I_i}_{-n_i}(1/q^+_0)\prod_j S^{a_j}_{-m_j}(1/q^+_0)\left\{\begin{array}{l} |I,q_0\rangle\,, \\ |\dot a,q_0\rangle\,, \end{array}\right. \tag{3.15}$$

where all the integers $\{n_i\}$ and $\{m_j\}$ are positive definite and the net momentum q is given by: $q^+ = q^+_0, q^- = \frac{N}{q^+_0} + q^-_0, q^I = q^I_0$, with $q^2_0 = 0$. N is the total level: $N = \sum_i n_i + \sum_j m_j$. The open string mass is given by: $M^2 = -\frac{q^2}{4} = \frac{N}{2}$. Therefore all the above DDF states are annihilated by both Q_B and L_0, hence are physical. They are also in one-to-one correspondence with the states in LCGS formalism. In defining the states in eqs. (3.15) one follows a particular canonical ordering of all the operators. Although according to the commutation relations (3.14) the states with different ordering are in general completely different states, we shall later see that effectively they will differ at most by an overall sign due to the reordering of the fermionic operators. We define a Hilbert space $\mathcal{H}_{\mathrm{DDF}}$ by the space spanned by the basis states (3.15). Next we define the conjugate basis states,

$$\left.\begin{array}{l} \langle\{(I_i,n_i)\},\{(a_j,m_j)\},I,q| \\ \langle\{(I_i,n_i)\},\{(a_j,m_j)\},\dot a,q| \end{array}\right\} \propto \left.\begin{array}{l} \langle I,q_0| \\ \langle\dot a,q_0| \end{array}\right\}\prod_j S^{a_j}_{m_j}(1/q^+_0)\prod_i A^{I_i}_{n_i}(1/q^+_0)\,. \tag{3.16}$$

Similar remarks about the ordering of the operators are in order in this case as well. If we can now argue that the basis defined this way is orthogonal such that by choosing the normalisation suitably the nonzero inner products can be written as,

$$\begin{aligned}
\langle\{(I_i,n_i)\},\{(a_j,m_j)\},I,q|\{(I'_i,n'_i)\},\{(a'_j,m'_j)\},I',q'\rangle &= \delta_{\{I_i,n_i\},\{I'_i,n'_i\}}\delta_{\{a_j,m_j\},\{a'_j,m'_j\}} \\
&\quad \delta_{I,I'}\delta(q^+ + q'^+)\delta(\vec{q}+\vec{q}')\,, \\
\langle\{(I_i,n_i)\},\{(a_j,m_j)\},\dot{a},q|\{(I'_i,n'_i)\},\{(a'_j,m'_j)\},\dot{a}',q'\rangle &= \delta_{\{I_i,n_i\},\{I'_i,n'_i\}}\delta_{\{a_j,m_j\},\{a'_j,m'_j\}} \\
&\quad \delta_{\dot{a},\dot{a}'}\delta(q^+ + q'^+)\delta(\vec{q}+\vec{q}')\,,
\end{aligned} \tag{3.17}$$

then it would imply that the DDF operators have the desired algebra in $\mathcal{H}_{\rm DDF}$.

$$\begin{aligned}
\left[A^I_m(1/q^+_0), A^J_n(1/q^+_0)\right]_{\mathcal{H}_{\rm DDF}} &= \delta^{IJ}\delta_{m,-n}\,, \\
\left\{S^a_m(1/q^+_0), S^b_n(1/q^+_0)\right\}_{\mathcal{H}_{\rm DDF}} &= \delta^{ab}\delta_{m,-n}\,, \\
\left[A^I_m(1/q^+_0), S^a_n(1/q^+_0)\right]_{\mathcal{H}_{\rm DDF}} &= 0\,.
\end{aligned} \tag{3.18}$$

Certainly the validity of our construction crucially relies on the orthogonality of the DDF states defined in this section. We shall prove it section 3.3.

Before going into the DDF construction for the anti-periodic sector let us explain the normalisation chosen in eqs. (3.4). Using the second equation in each of (3.10) and (B.9) one shows,

$$S^a_0 = \frac{1}{\sqrt{-i\sqrt{2}\alpha^+_0}} Q^a_L\,, \qquad \left\{S^a_0, S^b_0\right\} = \delta^{ab}\,, \tag{3.19}$$

which are exact in θ expansions and as desired for the fermionic zero modes in LCGS formalism [18]. Then using the first two equations in (3.5) one shows,

$$S^a_0 \begin{pmatrix} |I,q\rangle \\ |\dot{a},q\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma^I_{a\dot{a}} \begin{pmatrix} |\dot{a},q\rangle \\ |I,q\rangle \end{pmatrix}\,, \tag{3.20}$$

which is also expected.

3.2 Anti-periodic sector

Let us now turn to the construction of DDF states in the anti-periodic sector. Supersymmetry, which has played a crucial role in such construction in the periodic sector, is broken in this case. Nevertheless we shall now see that the non-supersymmetric open string spectrum can be obtained by a simple generalisation of the previous construction. In the anti-periodic sector any space-time fermion has half-integer modes whereas all the space-time bosons have the same integer mods as in the previous case. This means that the construction of the bosonic oscillators $A^I_n(1/q^+)$ still goes through with the commutation relation as given in (3.18). We define the half-integer fermionic modes in the following way,

$$S^a_r(1/q^+) = \frac{1}{\sqrt{-i\sqrt{2}\alpha^+_0}} \oint \frac{dz}{2\pi i}\mathcal{F}^a_L(k^- = -r/q^+, z)\,, \tag{3.21}$$

where $r \in \mathbf{Z} + 1/2$. In the present sector the vacuum $|\sigma\rangle$ is twisted so that it produces branch cut for the space-time fermions. The action of $S^a_r(1/q^+)$ is well defined on any excited sate of momentum q on this vacuum, because the branch cut in the space-time fermions is cancelled by the branch cut produced by the half-integer units of momentum in the DDF vertex $\mathcal{F}^a_L(k^- = -r/q^+, z)$. Using the second equation in (B.6) and the anti-periodicity of the fermions one can argue that these operators are BRST exact on any state of momentum q,

$$\left\{Q_B, S^a_r(1/q^+)\right\} = 0 \ . \tag{3.22}$$

The physical Hilbert space $\mathcal{H}_{\rm DDF}$ is now defined to be expanded by the following ghost number one states,

$$|\{(I_i, n_i)\}, \{(a_j, r_j)\}, q\rangle \propto \prod_i A^{I_i}_{-n_i}(1/q_0^+) \prod_j S^{a_j}_{-r_j}(1/q_0^+)|\sigma, q_0\rangle \, , \tag{3.23}$$

with fixed q_0^+ such that $q_0^2 = 1$. The net momentum q is now given by, $q^+ = q_0^+$, $q^- = \frac{N}{q_0^+} + q_0^-$, $q^I = q_0^I$, where the net level is: $N = \sum_i n_i + \sum_j r_j$. The open string mass is now given by: $M^2 = \frac{1}{2}\left(N - \frac{1}{2}\right)$. Proceeding similarly as in the periodic case by defining the conjugate states one argues the following commutation relations in the physical Hilbert space,

$$\left\{S^a_r(1/q^+), S^b_s(1/q^+)\right\}_{\mathcal{H}_{\rm DDF}} = \delta^{ab}\delta_{r,-s} \, , \qquad \left[A^I_n(1/q^+), S^a_r(1/q^+)\right]_{\mathcal{H}_{\rm DDF}} = 0 \ . \tag{3.24}$$

Again one needs the DDF basis to be orthogonal, an issue that will be discussed in the next section.

3.3 Validity of the construction

Validity of the DDF construction as done above relies on the orthogonality of the DDF states both in the periodic and anti-periodic sectors. Here this orthogonality will be proved. We shall, for definiteness, consider the periodic sector, generalisation to the anti-periodic sector being obvious.

We first notice, with the relation (3.20) in mind, that an arbitrary inner product between that states in (3.15) and (3.16) can be given the following form,

$$\mathcal{I} = \langle J| \prod_i S^{b_i}_{\bar{m}_i} \prod_j A^{J_j}_{\bar{n}_j} \prod_k A^{I_k}_{-n_k} \prod_l S^{a_l}_{-m_l}|I\rangle \, , \tag{3.25}$$

if we allow the zero modes for the fermionic oscillators. To reduce clutter, we have suppressed the momentum specification. The indices i, j, k, l run up to arbitrary positive integers. The bosonic mode numbers $\bar{n}_j, n_k$ are positive definite while the fermionic ones $\bar{m}_j, m_l$ are positive, including zero. Next we notice that, due to the θ-charge conservation, the matrix element of any operator of nonzero θ-charge between two vector ground states is zero. The only non-trivial operators that can have non-zero matrix elements are rotation generators which have zero θ-charge. Therefore after expanding all the operators in (3.25) in powers of θ the only terms that give non-zero results are those for which sum of the θ-charges of all the operators in a given product is zero. Here is a special example of a term that is potentially non-zero:

$$\mathcal{I}_{\text{special}} = \langle J| \prod_i S^{(\bar{o}_i)b_i}_{\bar{m}_i} \prod_j A^{(0)J_j}_{\bar{n}_j} \prod_k A^{(0)I_k}_{-n_k} \prod_l S^{(o_l)a_l}_{-m_l} |I\rangle \,,$$

$$\bar{o}_i, o_l = \pm 1 \,, \qquad \left(\sum_i \bar{o}_i + \sum_l o_l \right) = 0 \,, \tag{3.26}$$

where the extra index in the parenthesis refers to the θ-charge of the term in the θ-expansion of the relevant operator. We shall see later in what sense this inner product is special. Let us first try to compute the inner product using the following commutation relations,

$$\begin{aligned}
\left[A^{(0)I}_m(1/q_0^+), A^{(0)J}_n(1/q_0^+) \right] &= m\delta^{IJ}\delta_{m,-n} \,, \\
\left[A^{(0)I}_m(1/q_0^+), S^{(-1)a}_n(1/q_0^+) \right] &= 0 \,, \\
\left\{ S^{(-1)a}_m(1/q_0^+), S^{(-1)b}_n(1/q_0^+) \right\} &= 0 \,, \\
\left\{ S^{(-1)a}_m(1/q_0^+), S^{(1)b}_n(1/q_0^+) \right\} &= \frac{1}{2}\delta^{ab}\delta_{m,-n} \;.
\end{aligned} \tag{3.27}$$

The commutation relations involving $S^{(-1)}$'s (to simplify notation we are suppressing the indices that are not relevant for our discussion) guarantee that in order for the inner product to be non-zero we should have the following condition satisfied: let n^+ and n^- be the number of positively modded fermionic operators with θ-charge $+1$ and -1 respectively, then the number of negatively modded fermionic operators with θ-charge $+1$ and -1 are given by n^- and n^+ respectively. One can then move $S^{(-1)}$'s towards right or left, as appropriate, to absorb all the $S^{(1)}$'s. This way one gets rid of all the fermionic operators. Then the inner product of the bosonic operators can easily be found by using the first equation in (3.27). Therefore the final result should be,

$$\mathcal{I}_{\text{special}} \propto \delta_{\{J_j,\bar{n}_j\},\{I_k,\bar{n}_k\}}\delta_{\{b_i,\bar{m}_i\},\{a_l,m_l\}}\delta_{IJ} \;. \tag{3.28}$$

Certainly there is an obvious delta function involving momenta, which is suppressed in the above expression. One does not have other symmetry terms originating from interchange of the operators as the basis states have been defined with an ordering.

We shall now argue that the only terms that are non-zero in $\mathcal{I}$ are of the type $\mathcal{I}_{\text{special}}$. $\mathcal{I}_{\text{special}}$ is the kind of terms in $\mathcal{I}$ that come with the minimum number of $S^{(-1)}$'s. All the other terms with zero total θ-charge can be obtained by replacing some of the operators (both bosonic and fermionic) in $\mathcal{I}_{\text{special}}$ by the corresponding ones with higher θ-charge and balancing the total θ-charge by adding suitable number of extra $S^{(-1)}$'s. More we bring in higher θ-charge operators, bigger we make the mismatch between the numbers of $S^{(-1)}$'s and $S^{(1)}$'s. If the higher θ-charge operators commute with $S^{(-1)}$'s then the result will be zero. But generically this will not be the case. The final result can still be zero if the collection of all the higher θ-charge operators are unable to absorb all the extra $S^{(-1)}$'s through commutators. The necessary and sufficient condition for this to happen is the fact that the term in either of the DDF vertex operators $\mathcal{B}^I(k^-, z)$ and $\mathcal{F}_L^a(k^-, z)$ at the n-th order in θ-expansion does not have a term with charge $(n, 0)$ for $n > 1$. According to our notation, a term with charge (p, q) has left (θ_L) and right (θ_R) moving θ-charges p and q respectively. The above requirement is satisfied due to a theorem that we call *absence of maximal left moving θ-charges* which is stated and proved in appendix C. This establishes the fact that the inner product in eq. (3.25) is proportional to the right hand side of (3.28) and therefore the DDF basis states are orthogonal.

4. Physical components of boundary states for instantonic D-branes

A particular approach of studying open string boundary conditions and D-brane boundary states have been considered in [11] where one writes down the boundary conditions and boundary states in the free CFT by relaxing the pure spinor constraint. These boundary conditions and boundary states are easy to construct and are in one-to-one correspondence with the actual boundary conditions and boundary states of the constrained CFT. The boundary conditions in the free CFT produce the correct reflection property between the holomorphic and anti-holomorphic parts of any bulk insertion that is allowed in the pure spinor CFT. With suitable choice of vertex operators the boundary states are expected to produce correct results for all the closed string one-point functions of the actual theory. But these boundary states, as one might already expect, are not suitable for computation of the cylinder diagram. The reason is two-fold which we list below:

1. Having been constructed in a bigger Hilbert space, these boundary states contain degrees of freedom which do not belong to the actual theory. Let us call them *unphysical* degrees of freedom.

2. The boundary states have been constructed in the gauge unfixed theory.

The first problem could be solved simply by throwing away all the *unphysical* degrees of freedom. A covariant pure spinor boundary state at ghost number $(1, 1)$ can be defined in the following way:

$$|\mathrm{B}\rangle_{PS} = \sum_{i_{PS}} \varphi^{(B)}_{i_{PS}} |i_{PS}\rangle \,, \tag{4.1}$$

where $\{|i_{PS}\rangle\}$ is a complete basis of ghost number $(1, 1)$ states in pure spinor formalism (with the constraint imposed). The one point functions $\varphi^{(B)}_{i_{PS}}$ can be computed following the prescription of [11] using the boundary state $|\mathrm{B}\rangle_{\text{free}}$ constructed in the free CFT. One might think that the projected boundary state $|\mathrm{B}\rangle_{PS}$ could be evolved by world-sheet time evolution to compute the cylinder diagram. As argued and demonstrated explicitly through the long range force computation in [11], this is not true as the boundary sate still includes gauge degrees of freedom. In NSR formalism these gauge degrees of freedom are removed by a simple gauge fixation (Siegel gauge). It is in this particular gauge the closed string propagator in Schwinger parametrisation has an interpretation of world-sheet time evolution. It is not clear how to achieve this in pure spinor formalism. In summary, the problem is to find a suitable further projection of the boundary state $|B\rangle_{PS}$ to remove the gauge degrees of freedom so that the projected boundary state can be evolved by the world-sheet time evolution. Here we shall achieve this by projecting the boundary states onto the physical Hilbert space $\mathcal{H}_{\rm DDF}$ constructed in the closed string sector. Since the DDF construction gives explicit expression for the LCGS variables in terms of the pure spinor variables, one would expect that the projected boundary states would take the same form of the LCGS boundary states in terms of the DDF operators. We shall see that this expectation is actually correct. Before going into the further details, we should mention that because of the special kinematical condition that q_0^+ is fixed and non-zero for all the DDF states, these are suitable to extract the physical components of only the instantonic boundary states which impose Dirichlet boundary conditions along both the light-cone directions. For a lorentzian D-brane having Neumann boundary condition along both the light-cone directions we need states which have both $q_0^\pm$ to be zero. For D-branes which have Neumann boundary condition along one of the light-cone directions and Dirichlet along the other q_0^+ needs to vary over the allowed states. We have discussed in appendix (D) the boundary conditions for the instantonic D-branes in the free CFT, following the same approach of [11].

The DDF states in the closed string theory can be constructed simply by constructing the DDF operators $A_n^I(1/q_0^+)$, $S_n^a(1/q_0^+)$ and $\tilde{A}_n^I(1/q_0^+)$, $\tilde{S}_n^a(1/q_0^+)$, as in the previous section, in the left and right moving sectors separately. Then we define, as before, the DDF states which may be denoted as,

$$\begin{pmatrix} |\{I_i,n_i\},\{a_j,m_j\},I,q\rangle_L \\ |\{I_i,n_i\},\{a_j,m_j\},\dot{a},q\rangle_L \end{pmatrix} \otimes \begin{pmatrix} |\{\tilde{I}_i,\tilde{n}_i\},\{\tilde{a}_j,\tilde{m}_j\},\tilde{I},q\rangle_R \\ |\{\tilde{I}_i,\tilde{n}_i\},\{\tilde{a}_j,\tilde{m}_j\},\tilde{\dot{a}},q\rangle_R \end{pmatrix}, \tag{4.2}$$

with $N = \tilde{N}$, which comes, as usual, from the $L_0 = \tilde{L}_0$ constraint. The above states correspond to all the physical degrees of freedom and are on-shell with mass given by: $M^2 = 2N$. Given the ghost number $(1,1)$ covariant boundary state $|\mathrm{Inst}\rangle_{PS}$ of an instantonic D-brane in pure spinor formalism, its physical component is given by,

$$|\mathrm{Inst}, q^+\rangle_{\rm phys} = \sum_i |i,q\rangle\langle i,q|\mathrm{Inst}\rangle_{PS} = \sum_i \varphi_i^{(Inst)}|i,q\rangle\,, \tag{4.3}$$

where the states $|i,q\rangle$ are the ghost number $(1,1)$ orthonormal basis states in $\mathcal{H}_{\rm DDF}$ given in (4.2) with a fixed q^+ and the states $\langle i,q^+|$ are the corresponding conjugate states with ghost number $(2,2)$. The sum over i in the above equation includes integration over spatial components of momentum as well as the discrete levels. The coefficients $\varphi_i^{(Inst)}$ can be computed using the boundary states $|\mathrm{Inst}\rangle_{\rm free}$ constructed in the free theory following the prescription of [11]. Therefore given $|\mathrm{Inst}\rangle_{\rm free}$, $|\mathrm{Inst},q^+\rangle_{\rm phys}$ can be constructed unambiguously. But to get a closed form expression for $|\mathrm{Inst},q^+\rangle_{\rm phys}$ we shall proceed in a less direct way. SO(8) covariant boundary states for the BPS and non-BPS instantonic D-branes in LCGS formalism are already known [12, 14, 15]. The pure spinor boundary state in (4.3) is expected to take the same form as the corresponding one in LCGS formalism. We shall argue that this is in fact true by deriving the gluing conditions satisfied by the DDF operators on $|\mathrm{Inst},q^+\rangle_{\rm phys}$ and showing that they are same as the corresponding ones in LCGS formalism.

The DDF gluing conditions will be obtained by using boundary conditions written in open string channel and then converting that to the closed string channel as needed. Using the mode expansion of X^+ one can argue that in the closed string channel at $\tau = 0$,

$$\left[e^{ink_0 X_L^+}(e^{i\sigma}) - e^{-ink_0 X_R^+}(e^{-i\sigma})\right] |\mathrm{Inst}, q^+\rangle_{\rm phys} = 0\,, \tag{4.4}$$

which can be used, in addition to the boundary condition in eq. (D.3), to argue that for an instantonic BPS Dp-brane one has,

$$\left.\begin{array}{c} \left[\mathcal{B}^I(-nk_0, e^{i\sigma}) - e^{-2i\sigma}(\mathcal{M}^V)^I{}_J \tilde{\mathcal{B}}^J(nk_0, e^{-i\sigma})\right] \\ \left[\mathcal{F}_L^a(-nk_0, e^{i\sigma}) + i\eta e^{-2i\sigma}\mathcal{M}^S_{ab}\tilde{\mathcal{F}}_L^b(nk_0, e^{-i\sigma})\right] \end{array}\right\} |\mathrm{Inst}_p, \eta, q^+\rangle_{\rm phys} = 0\,, \tag{4.5}$$

where $\mathcal{M}^V$ is the 8-dimensional block of M^V (as defined below eq. (D.1)) corresponding to the light-cone transverse directions. We define matrices $\mathcal{M}^S$ and $\mathcal{M}^C$ in the following way,

$$M^S = \bar{M}^S = \begin{pmatrix} \mathcal{M}^S_{ab} & 0 \\ 0 & \mathcal{M}^C_{\dot{a}\dot{b}} \end{pmatrix}, \tag{4.6}$$

where the matrices M^S and $\bar{M}^S$ are defined in eqs(D.4). Upon recalling the definitions (3.10), eqs. (4.5) readily give the following gluing conditions for the DDF operators.

$$\left.\begin{array}{c} \left[A_n^I(1/q^+) - (\mathcal{M}^V)_{IJ}\tilde{A}^J_{-n}(1/q^+)\right] \\ \left[S_n^a(1/q^+) + i\eta\mathcal{M}^S_{ab}\tilde{S}^b_{-n}(1/q^+)\right] \end{array}\right\} |\mathrm{Inst}_p, \eta, q^+\rangle_{\rm phys} = 0\,, \qquad \forall n \in \mathbf{Z}\,. \tag{4.7}$$

The bosonic part of the gluing conditions satisfied by a non-BPS D-instanton takes the same form as in (4.7). To obtain the fermionic part we proceed as follows: Writing the integrated gluino vertex operator in the following form: $\mathcal{F}_\alpha(k,z) = \mathcal{G}_\alpha(z)e^{ik.X}(z)$, we first use covariance to argue that $\mathcal{G}_\alpha(z)$ satisfies the same boundary condition as $p_\alpha(z)$ as given by the last equation in (2.1). The SO(8) decomposition of this boundary condition gives on UHP,

$$\mathcal{G}_L^a(z)\mathcal{G}_L^b(w) = -\mathcal{M}^{ab}_{cd}\,\tilde{\mathcal{G}}_L^c(\bar{z})\tilde{\mathcal{G}}_L^d(\bar{w})\,, \qquad \text{at } z = \bar{z}\,, w = \bar{w}\,, \tag{4.8}$$

where the coupling matrix $\mathcal{M}^{ab}_{cd}$ is given by,

$$\mathcal{M}^{ab}_{cd} = \frac{1}{8}\delta_{ab}\delta_{cd} + \frac{1}{16}\sum_{I,J}\lambda_I\lambda_J\sigma^{IJ}_{ab}\sigma^{IJ}_{cd} + \frac{1}{192}\sum_{\{IJKL\}\in\mathcal{K}^{(4)}}\lambda_I\lambda_J\lambda_K\lambda_L\sigma^{IJKL}_{ab}\sigma^{IJKL}_{cd}\,, \tag{4.9}$$

where we have used $(\mathcal{M}^V)^I{}_J = \lambda_I\delta^{IJ}$ for notational simplicity and $\mathcal{K}^{(4)}$ is defined, analogously to $\mathcal{K}^{(5)}$ in eqs. (2.3), for sets of four integers instead of five. Using (4.8) and (4.4) one can argue that the following condition is satisfied for a non-BPS instantonic Dp-brane in the closed string channel at $\tau = 0$,

$$\left[\mathcal{F}_L^a(-mk_0, e^{i\sigma})\mathcal{F}_L^b(-nk_0, e^{i\sigma'}) + e^{-2i(\sigma+\sigma')}\mathcal{M}^{ab}_{cd}\tilde{\mathcal{F}}_L^c(mk_0, e^{i\sigma})\mathcal{F}_L^d(nk_0, e^{i\sigma'})\right]|\mathrm{Inst}_p, q^+\rangle_{\rm phys} = 0, \tag{4.10}$$

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which implies the following gluing condition for the fermionic DDF operators,

$$\left[S^a_m(1/q^+)S^b_n(1/q^+) + \mathcal{M}^{ab}_{cd}\tilde{S}^c_{-m}(1/q^+)\tilde{S}^d_{-n}(1/q^+)\right]|\text{Inst}_p, q^+\rangle_{\text{phys}} = 0\,, \qquad \forall m,n \in \mathbf{Z}\,. \tag{4.11}$$

Eqs. (4.7) and (4.11) are precisely the same gluing conditions satisfied by the BPS and non-BPS instantonic D-brane boundary states in LCGS formalism as discussed in [12] and [15] respectively. The physical components of the D-instanton boundary states in pure spinor formalism can therefore be found simply by replacing the LCGS oscillators by the corresponding DDF operators constructed here in the expressions for the boundary states found in [12] and [15] (with the obvious change of notations for the SO(8) vector and spinor matrices). Notice that the physical components of the D-instanton boundary states constructed this way have very complicated expressions in terms of the pure spinor variables as the DDF operators have θ-expansions. But for computations restricted to $\mathcal{H}_{\text{DDF}}$ these states behave as simply as the boundary states in LCGS formalism.

5. The cylinder diagram

Having removed all the unphysical degrees of freedom which one should not let propagate in the cylinder diagram we can now evolve the projected boundary state $|\text{Inst}, q^+\rangle_{\text{phys}}$ by the closed string propagator $1/(L_0 + \tilde{L}_0)$.

$$\mathcal{C}(X^+, X^-) \propto \int dq^+ dq^- \langle\text{Inst}, -q^-, -q^+|\frac{e^{iq^+X^- + iq^-X^+}}{L_0 + \tilde{L}_0}|\text{Inst}', q^-, q^+\rangle\,, \tag{5.1}$$

where $X^\pm$ are the separation between the two branes along the light-cone directions. There is also a separation in the transverse direction which we have suppressed. The states have been allowed arbitrary q^- as required by the Fourier transform of the position eigen states. But we shall see that the propagating states will have the on-shell value. Writing $(L_0 + \tilde{L}_0)$ in the following form,

$$L_0 + \tilde{L}_0 = -2p^+(p^- - H)\,, \tag{5.2}$$

where,

$$H = \frac{1}{2p^+}(\vec{p}^2 + N + \tilde{N})\,, \tag{5.3}$$

with $N = N^{(X)} + N^{(p,\theta)} + N^{(w,\lambda)}$ (similarly for the right moving sector) and following the same steps as in [20] one arrives at the following expression for the cylinder diagram,

$$\mathcal{C}(X^+, X^-) \propto \int_0^\infty \frac{d\tau}{\tau} e^{\frac{iX^+X^-}{2\pi\tau}} \langle\text{Inst}, -q^+|e^{i\pi\tau(\vec{p}^2 + N + \tilde{N})}|\text{Inst}', q^+\rangle\,, \tag{5.4}$$

where $\tau = \frac{X^+}{2\pi q^+}$ can be easily identified with the modulus of the lorentzian cylinder. Going to the euclidean world-sheet by the Wick rotation: $\tau \to it$, one arrives at,

$$\mathcal{C}(X^+, X^-) \propto \int_0^\infty \frac{dt}{t} e^{\frac{X^+X^-}{2\pi t}} \langle\text{Inst}, -q^+|e^{-\pi t(\vec{p}^2 + N + \tilde{N})}|\text{Inst}', q^+\rangle\,. \tag{5.5}$$

Computation of this quantity is well-understood and the open-closed duality is manifest in the result.

6. Conclusion

The open string boundary conditions and boundary states for both BPS and non-BPS D-branes in pure spinor formalism were written down in [11] in the unconstrained CFT by relaxing the pure spinor constraint. It was argued that these boundary conditions and boundary states are suitable to compute boundary conformal field theory correlators and closed string one point functions respectively. But one can not evolve these boundary states according to the world-sheet time evolution to compute the cylinder diagram with manifest open-closed duality. This is because one does not know how to remove the gauge degrees of freedom propagating along the cylinder. The cylinder diagram can still be computed if one is able to project these boundary states onto a physical Hilbert space free of such degrees of freedom.

In this paper we have explicitly constructed such a physical Hilbert space in pure spinor formalism. By exploiting the manifest supersymmetry of the formalism, this Hilbert space has been constructed by performing a supesymmetric version of the usual DDF construction [16]. This gives an explicit realisation of all the states obtained in LCGS formalism. The validity of our construction has been justified by proving that the DDF operators constructed here have the same commutation relations as those of the LCGS oscillators in the physical Hilbert space. Outside this Hilbert space the commutators have non-trivial θ-expansions.

The DDF construction for open strings on BPS D-branes (closed strings) implicitly defines the ghost number one (two) unintegrated vertex operators for all the string states in the BRST cohomology with special kinematical conditions.[7] All these vertex operators take a form where the ghost number of the operator is contributed only by the zero modes of the pure spinor ghosts. Using the boundary conditions in [11] it is argued that there are two sectors of open strings on a non-BPS D-brane: periodic (R) and anti-periodic (NS). The analysis for the R sector goes in the same way as that corresponding to the BPS D-branes. Although the DDF construction for the NS sector is well defined, the unintegrated vertex operators, that are needed for the scattering amplitude computations, can not be derived from this construction unless the vertex operator for the unique ground state, which represents the open string tachyon, is understood.[8] Understanding of how to explicitly construct this ground state is an interesting and important open question. This is an example of a more generic question of how to construct the ground states for open strings stretching between branes at angles which will allow more general boundary conditions.

Going back to the discussion of boundary states, we derive the gluing conditions for the DDF operators satisfied on the boundary states for both the BPS and non-BPS instantonic D-branes. We show that these conditions are exactly the same as those satisfied by the oscillators in LCGS formalism [12, 15]. Therefore the projected boundary states in pure spinor formalism can be obtained simply by replacing the LCGS oscillators by the DDF operators constructed here in the expressions for the boundary states written down

[7]See [22] for computation of the covariant vertex operators at the first massive level.

[8]I thank N. Berkovits for discussion on this point.

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in [12, 15]. This construction offers an explicit embedding of all the computations that can possibly be done in LCGS formalism into pure spinor formalism, a particular example being the cylinder diagram with manifest open-closed duality. However, computing the cylinder diagram using a covariant boundary state still remains an open question. It is important to identify the relevant covariant basis for which the techniques of [23] may prove useful.

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A. Notation and convention

We follow the same notation and convention for the 32 and 16 dimensional gamma matrices as given in [11]. Therefore all the gamma matrix properties and Fiertz identities summarised in the relevant appendix of [11] still hold. Here we shall consider an explicit SO(8) decomposition. We define the light-cone components $A^{\pm}$ of a 10-dimensional vector A^{μ} in the following way,

$$A^{\pm} = \frac{1}{\sqrt{2}}(A^0 \pm A^9) \; . \tag{A.1}$$

A 16-dimensional chiral spinor ξ of either chirality is decomposed into the left and right moving SO(8) spinors in the following way,

$$\xi = (\xi_L^a ,\; \xi_R^{\dot{a}}) \; . \tag{A.2}$$

The 16-dimensional gamma matrices of [11] are given by,

$$\gamma^0 = -\bar{\gamma}^0 = \mathbb{I}_{16} \, , \qquad \gamma^I = \bar{\gamma}^I = \begin{pmatrix} 0 & \sigma^I_{a\dot{a}} \\ \bar{\sigma}^I_{\dot{a}a} & 0 \end{pmatrix} , \qquad \gamma^9 = \bar{\gamma}^9 = \gamma^1\gamma^2 \cdots \gamma^8 = \begin{pmatrix} \mathbb{I}_8 & 0 \\ 0 & -\mathbb{I}_8 \end{pmatrix} , \tag{A.3}$$

where $\bar{\sigma} = (\sigma)^T = \sigma = \sigma^*$.

We shall now collect some of the expressions that are relevant for open strings and are directly needed for the computation of the present paper. We work in the $\alpha' = 2$ unit, such that,

$$X^{\mu}(z) X^{\nu}(w) \sim -\eta^{\mu\nu} \log |z - w| \; . \tag{A.4}$$

The supersymmetry charge is given by,

$$\begin{aligned} Q_{\alpha} &= \oint \frac{dz}{2\pi i} q_{\alpha}(z) \, , \\ q_{\alpha} &= p_{\alpha} + \frac{1}{2} \left(\bar{\gamma}^{\mu}\theta\right)_{\alpha} \partial X_{\mu} + \frac{1}{24} \left(\bar{\gamma}^{\mu}\theta\right)_{\alpha} \left(\theta\bar{\gamma}_{\mu}\partial\theta\right) \; . \end{aligned} \tag{A.5}$$

Using (A.4) and,

$$p_\alpha(z)\theta^\beta(w) \sim \frac{\delta_\alpha{}^\beta}{z-w}\,, \tag{A.6}$$

one can derive the following supersymmetry algebra,

$$\{Q_\alpha, Q_\beta\} = \bar\gamma^\mu_{\alpha\beta}\oint \frac{dz}{2\pi i}\partial X_\mu(z)\,, \tag{A.7}$$

which takes the following form in the SO(8) notation,

$$\begin{aligned}\left\{Q^a_L, Q^b_L\right\} &= \sqrt{2}\delta^{ab}\oint \frac{dz}{2\pi i}\partial X^+(z)\,,\\ \{Q^a_L, Q^{\dot a}_R\} &= \sigma^I_{a\dot a}\oint \frac{dz}{2\pi i}\partial X^I(z)\,,\\ \left\{Q^{\dot a}_R, Q^{\dot b}_R\right\} &= \sqrt{2}\delta^{\dot a\dot b}\oint \frac{dz}{2\pi i}\partial X^-(z)\ .\end{aligned} \tag{A.8}$$

The BRST charge is given by,

$$\begin{aligned}Q_B &= \oint \frac{dz}{2\pi i} q_B(z)\,,\\ q_B &= \lambda^\alpha(z) d_\alpha(z)\,, \qquad d_\alpha = p_\alpha - \frac{1}{2}(\bar\gamma^\mu\theta)_\alpha\partial X_\mu - \frac{1}{8}(\bar\gamma^\mu\theta)_\alpha(\theta\bar\gamma_\mu\partial\theta)\ .\end{aligned} \tag{A.9}$$

The fermionic matter and the pure spinor ghost contributions to the Lorentz currents $M^{\mu\nu}(z)$ and $N^{\mu\nu}(z)$ respectively are given by,

$$M^{\mu\nu} = -\frac{1}{2}(p\gamma^{\mu\nu}\theta)\,, \qquad N^{\mu\nu} = \frac{1}{2}(w\gamma^{\mu\nu}\lambda)\ . \tag{A.10}$$

They form SO(9,1) current algebra at levels 4 and -3 respectively and satisfy the following OPE's,

$$\begin{aligned}M^{\mu\nu}(z)\theta^\alpha(w) &\sim \frac{1}{2(z-w)}\left(\gamma^{\mu\nu}\theta(w)\right)^\alpha\,, \qquad M^{\mu\nu}(z)p_\alpha(w) \sim \frac{1}{2(z-w)}\left(\bar\gamma^{\mu\nu}p(w)\right)_\alpha\,,\\ N^{\mu\nu}(z)\lambda^\alpha(w) &\sim \frac{1}{2(z-w)}\left(\gamma^{\mu\nu}\lambda(w)\right)^\alpha\ .\end{aligned} \tag{A.11}$$

Finally we define the Virasoro zero mode in the following way:

$$L_0 = \frac{\alpha_0^2}{2} + N^{(X)} + N^{(p,\theta)} + N^{(w,\lambda)} + a\,, \tag{A.12}$$

where $N^{(X)}$, $N^{(p,\theta)}$ and $N^{(w,\lambda)}$ are the level operators for the bosonic matter, fermionic matter and bosonic ghost sectors respectively. $N^{(X)}$ and $N^{(p,\theta)}$ are defined in the usual way. For the ghost sector this may be defined through the following commutation relations:

$$[N^{(w,\lambda)}, \lambda^\alpha_r] = -r\lambda^\alpha_r\,, \qquad [N^{(w,\lambda)}, N^{\mu\nu}_n] = -nN^{\mu\nu}_n\,, \qquad [N^{(w,\lambda)}, J_n] = -nJ_n\,, \tag{A.13}$$

where n is an integer and r is an integer or half integer depending on whether we are considering the periodic or anti-periodic sector respectively. $N^{\mu\nu}_n$ and J_n are the modes of the currents $N^{\mu\nu}(z)$ and $J(z) \propto w_\alpha(z)\lambda^\alpha(z)$ respectively. The normal ordering constant a in eq. (A.12) is given by,

$$a = \begin{cases} 0\,, & \text{periodic sector,} \\ -\frac{1}{2}\,, & \text{anti-periodic sector .} \end{cases} \tag{A.14}$$

The value for the periodic sector is easily understood from the fact that we have a Bose-Fermi degeneracy in this sector. For the anti-periodic sector we have fixed this by requiring a physical condition, namely the open-closed duality. In this paper we have constructed the physical Hilbert space of LCGS formalism explicitly in terms of the pure spinor variables. The projected boundary states onto this Hilbert space are suitable for the computation of the cylinder diagram with manifest open-closed duality. There is no ambiguity in the computation on the closed string side. Hence it gives a unique answer which has to be consistent with the open string channel computation. This fixes the L_0 eigenvalue of the unique ground state in the NS sector.

B. Massless vertex operators and supersymmetry

Physical vertex operators in the unintegrated form are given by certain ghost number one operators in the BRST cohomology. The super-Poincaré invariant massless vertex operator is given by: $\lambda^\alpha(z)A_\alpha(X(z),\theta(z))$ where the function $A_\alpha(x,\theta)$ is the spinor potential for D=10, N=1 super-Maxwell theory satisfying $D_\alpha(\gamma^{\mu_1\cdots\mu_5})^{\alpha\beta}A_\beta = 0$ for any $\mu_1,\dots\mu_5$ and $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\bar{\gamma}^\mu_{\alpha\beta}\theta^\beta\partial_\mu$. The gauge invariance is given by: $\delta A_\alpha = D_\alpha\Omega$. Using this gauge invariance the massless vertex operators can be given the following form in momentum space,

$$u(k,z) = a_\mu(k)b^\mu(k,z) + \xi^\alpha(k)f_\alpha(k,z)\,, \tag{B.1}$$

where,

$$\begin{aligned} b^\mu(k,z) &= \frac{1}{2}\left(\lambda(z)\bar{\gamma}^\mu\theta(z)\right)e^{ik.X}(z) + \cdots\,, \\ f_\alpha(k,z) &= \frac{1}{3}\left(\lambda(z)\bar{\gamma}^\mu\theta(z)\right)\left(\bar{\gamma}_\mu\theta(z)\right)_\alpha e^{ik.X}(z) + \cdots\,, \end{aligned} \tag{B.2}$$

where the dots refer to terms higher order in θ. Notice that the gluon vertex operator $b^\mu(k,z)$ is world-sheet fermionic as it contains terms with odd θ-charges only. Gluino vertex operator $f_\alpha(k,z)$, on the other hand, has even θ-charge, hence is bosonic. Equation of motion and the residual gauge invariance are given by,

$$\begin{aligned} &k^2 = 0\,, \qquad k_\mu a^\mu(k) = 0\,, \qquad k_\mu\left(\bar{\gamma}^\mu\xi(k)\right)_\alpha = 0\,, \\ &\delta a^\mu(k) = \Lambda(k)k^\mu\,, \qquad \delta\xi^\alpha(k) = 0\,. \end{aligned} \tag{B.3}$$

The vertex operators in (B.2) are BRST closed when the above equations of motion are satisfied,

$$\{Q_B, b^\mu(k,z)\} = 0\,, \qquad [Q_B, f_\alpha(k,z)] = 0\,. \tag{B.4}$$

The on-shell supersymmetry transformations are (up to BRST exact terms),

$$\{Q_\alpha, b^\mu(k,z)\} = -\frac{i}{2} k_\nu \bar\gamma^{\mu\nu}{}_\alpha{}^\beta f_\beta(k,z)\,, \qquad [Q_\alpha, f_\beta(k,z)] = -\bar\gamma^\mu_{\alpha\beta} b_\mu(k,z)\ . \tag{B.5}$$

The integrated vertex operators have ghost number zero and satisfy the following relations with the unintegrated ones,

$$[Q_B, \mathcal{B}^\mu(k,z)] = \partial b^\mu(k,z)\,, \qquad \{Q_B, \mathcal{F}_\alpha(k,z)\} = \partial f_\alpha(k,z)\ . \tag{B.6}$$

The on shell supersymmetry transformations take the similar form as in eqs. (B.5) and are given, up to total derivative terms, by,

$$[Q_\alpha, \mathcal{B}^\mu(k,z)] = \frac{i}{2} k_\nu \bar\gamma^{\mu\nu}{}_\alpha{}^\beta \mathcal{F}_\beta(k,z)\,, \qquad \{Q_\alpha, \mathcal{F}_\beta(k,z)\} = \bar\gamma^\mu_{\alpha\beta}\mathcal{B}_\mu(k,z)\ . \tag{B.7}$$

Clearly $\mathcal{B}^\mu(k,z)$ and $\mathcal{F}_\alpha(k,z)$ have θ expansions with only even and odd order terms respectively. To justify our construction of the DDF operators we need explicit expressions for only up to first order terms.

$$\begin{aligned}
\mathcal{B}^\mu(k,z) &= \left(\partial X^\mu(z) + i k_\nu L^{\nu\mu}(z)\right) e^{ik.X}(z) + \cdots\,,\\
\mathcal{F}_\alpha(k,z) &= p_\alpha(z) e^{ik.X}(z)\\
&\quad + (\bar\gamma^\mu\theta(z))_\alpha \left[\frac{1}{2}\partial X_\mu(z) + i k^\nu \left(N_{\nu\mu}(z) + \frac{1}{2} M_{\nu\mu}(z)\right)\right] e^{ik.X}(z) + \cdots\,,
\end{aligned} \tag{B.8}$$

where $L^{\mu\nu} = M^{\mu\nu} + N^{\mu\nu}$ is the fermionic matter and pure spinor ghost contribution to the SO$(9,1)$ Lorentz current at level 1. This should be identified with the fermonic matter contribution to the Lorentz current in NSR formalism. Notice that at zero momentum θ-expansion of these operators simplify. Using necessary OPE's it can be argued that with,

$$\mathcal{B}^\mu(0,z) = \partial X^\mu(z)\,, \qquad \mathcal{F}_\alpha(0,z) = q_\alpha(z)\,, \tag{B.9}$$

one can satisfy both eqs. (B.6) and (B.4) with [21],

$$b^\mu(0,z) = \frac{1}{2}\left(\lambda(z)\bar\gamma^\mu\theta(z)\right)\,, \qquad f_\alpha(0,z) = \frac{1}{3}\left(\lambda(z)\bar\gamma^\mu\theta(z)\right)\left(\bar\gamma_\mu\theta(z)\right)_\alpha\ . \tag{B.10}$$

This result is crucial to show eqs. (3.19).

We shall now show that the first order term in the θ-expansion of $\mathcal{F}_\alpha(k,z)$ is as given in eq. (B.8). Writing,

$$\begin{aligned}
\mathcal{B}_\mu(k,z) &= \mathcal{B}^{(0)}_\mu(k,z) + \mathcal{B}^{(2)}_\mu(k,z) + \cdots\,,\\
\mathcal{F}_\alpha(k,z) &= \mathcal{F}^{(-1)}_\alpha(k,z) + \mathcal{F}^{(1)}_\alpha(k,z) + \cdots\,,\\
q_\alpha(z) &= q^{(-1)}_\alpha(z) + q^{(1)}_\alpha(z) + q^{(3)}_\alpha(z)\,,
\end{aligned} \tag{B.11}$$

with the integers appearing in the superscripts of various term referring to the θ-charge and using the supersymmetry transformations (B.7) one concludes (up to possible total derivative terms),

$$\mathrm{Res}_{z\to w}\left[q^{(-1)}_\alpha(z)\xi^\beta(k)\mathcal{F}^{(1)}_\beta(k,w) + q^{(1)}_\alpha(z)\xi^\beta(k)\mathcal{F}^{(-1)}_\beta(k,w)\right] = -\left(\bar\gamma^\mu\xi(k)\right)_\alpha \mathcal{B}^{(0)}_\mu(k,w)\,, \tag{B.12}$$

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where $\xi(k)$ satisfies the on-shell condition in (B.3). Reading out $q^{(1)}(z)$ and $\mathcal{F}_\alpha^{(-1)}(k,z)$ from eqs. (A.5) and (B.8) one shows,

$$\begin{aligned}\mathrm{Res}_{z\to w} q_\alpha^{(1)}(z)\xi^\beta(k)\mathcal{F}_\beta^{(-1)}(k,w) &= -\frac{1}{2}\left(\bar\gamma^\mu\xi(k)\right)_\alpha \partial X_\mu(w) e^{ik.X}(w) \\ &+\frac{i}{2}k_\mu\xi^\beta(k)\left(\bar\gamma^\mu\theta(w)\right)_\alpha p_\beta(w)e^{ik.X}(w)\ . \end{aligned} \tag{B.13}$$

Using the on-shell condition: $k_\mu\bar\gamma^\mu_{\alpha\beta}\xi^\beta(k)=0$ and the gamma matrix property: $\eta_{\mu\nu}\bar\gamma^\mu_{(\alpha\beta}\bar\gamma^\nu_{\gamma)\delta} = 0$, one can do a manipulation to write,

$$k_\mu\xi^\beta\left(\bar\gamma^\mu\theta\right)_\alpha p_\beta = k_\mu\left(\bar\gamma_\nu\xi\right)_\alpha M^{\nu\mu}+\frac{1}{2}k_\mu\left(\theta\bar\gamma_\nu\xi\right)\left(\bar\gamma^{\nu\mu}p\right)_\alpha\ . \tag{B.14}$$

Using this and reading out the expression for $\mathcal{B}_\mu^{(0)}(k,z)$ from eqs. (B.8) one can write,

$$\begin{aligned}\mathrm{Res}_{z\to w} p_\alpha(z)\xi^\beta(k)\mathcal{G}_\beta^{(1)}(k,w) &= -\frac{1}{2}\left(\bar\gamma^\mu\xi(k)\right)_\alpha\partial X_\mu(w) - i\left(\bar\gamma^\mu\xi(k)\right)_\alpha k^\nu N_{\nu\mu}(w) \\ &-\frac{i}{2}\left(\bar\gamma^\mu\xi(k)\right)k^\nu M_{\nu\mu}(w)+\frac{i}{4}\left(\theta(w)\bar\gamma^\mu\xi(w)\right)k^\nu\left(\bar\gamma_{\nu\mu}p\right)_\alpha\ . \end{aligned} \tag{B.15}$$

where we have written,

$$\mathcal{F}_\alpha^{(1)}(k,z)=\mathcal{G}_\alpha^{(1)}(k,z)e^{ik.X}(z)\ . \tag{B.16}$$

It can be explicitly checked that the expression for $\mathcal{G}_\alpha^{(1)}(z)$ as read from eqs. (B.16) and (B.8) indeed satisfies eq. (B.15). We should also check the consistency of this result with the BRST property. The following equation,

$$[Q_B^{(-1)},\xi^\alpha(k)\mathcal{F}_\alpha^{(1)}(k,z)]+[Q_B^{(1)},\xi^\alpha(k)\mathcal{F}_\alpha^{(-1)}(k,z)]=0\,, \tag{B.17}$$

which is obtained by expanding the second equation in (B.6) in powers of θ, needs to be satisfied. Reading out the expressions for $q_B^{(-1)}(z)$ and $q_B^{(1)}(z)$ from eq. (A.9) one first derives,

$$\begin{aligned}\mathrm{Res}_{z\to w} q_B^{(-1)}(z)\mathcal{F}_\alpha^{(1)}(k,w) &= \left(\bar\gamma^\mu\lambda(w)\right)_\alpha\left[\frac{1}{2}\partial X_\mu(w)+ik^\nu\left(N_{\nu\mu}(w)+\frac{1}{2}M_{\nu\mu}(w)\right)\right]e^{ik.X}(w) \\ &-\frac{i}{4}k^\nu\left(\bar\gamma^\mu\theta(w)\right)_\alpha\left(\lambda(w)\bar\gamma_{\nu\mu}p(w)\right)e^{ik.X}(w)\,, \\ \mathrm{Res}_{z\to w} q_B^{(1)}(z)\mathcal{F}_\alpha^{(-1)}(k,w) &= \Big[-\frac{1}{2}\left(\bar\gamma^\mu\lambda(w)\right)_\alpha\partial X_\mu(w)+ \\ &\frac{i}{2}k_\mu\left(\bar\gamma^\mu\partial\lambda(w)\right)_\alpha+\frac{i}{2}k_\mu\left(\lambda(w)\bar\gamma^\mu\theta(w)\right)p_\alpha(w)\Big]e^{ik.X}(w)\ . \end{aligned} \tag{B.18}$$

Then using the on-shell condition for $\xi^\alpha(k)$ and the result (B.14) one shows that the condition (B.17) is indeed satisfied. One may wonder what happens to the $N_{\mu\nu}$ dependent term in the first equation as there is no other term that can cancel it. This term can be shown to drop off by using the following identity,

$$N^{\mu\nu}\left(\bar\gamma_\nu\lambda\right)_\alpha=-\frac{1}{4}\left(w\gamma^\mu\bar\gamma^\nu\right)_\alpha\left(\lambda\bar\gamma_\nu\lambda\right)-\frac{1}{2}\left(\bar\gamma^\mu\lambda\right)_\alpha(w\lambda)\ , \tag{B.19}$$

and the on-shell condition for $\xi^\alpha(k)$.

C. The *Absence of maximal left-moving θ-charges* theorem

Below we state and prove the *absence of maximal left-moving θ-charges* theorem.

Theorem. *The n-th order terms $\mathcal{B}^{(n)I}(k^-, z)$ and $\mathcal{F}_L^{(n)a}(k^-, z)$ in the DDF vertex operators $\mathcal{B}^I(k^-, z)$ and $\mathcal{F}_L^a(k^-, z)$ respectively do not contain terms with charge $(n, 0)$ for $n > 1$ and $(n+1, -1)$ for $n > -1$.*

Proof. We start by proving that $\mathcal{O}^{(n)}(z)$ does not have a term with charge $(n+1, -1)$ for $n > -1$, where $\mathcal{O}^{(n)}(z)$ stands for either $\mathcal{B}^{(n)I}(k^-, z)$ or $\mathcal{F}_L^{(n)a}(k^-, z)$. To do that let us consider the two commutation relations involving Q_L^a in eqs. (3.7) and (3.8) for non-zero k^-. These relations imply,

$$\mathrm{Res}_{z\to w}\left[q_L^{(-1,0)}(z)\mathcal{O}^{(n)}(w) + \left\{q_L^{(0,1)}(z) + q_L^{(1,0)}(z)\right\}\mathcal{O}^{(n-2)}(w) + q_L^{(1,2)}(z)\mathcal{O}^{(n-4)}(w)\right] = 0\,, \tag{C.1}$$

where,

$$q_L^{(-1,0)} = p_L\,, \qquad q_L^{(0,1)} = \frac{1}{2}\partial X^I\sigma^I\theta_R\,, \qquad q_L^{(1,0)} = \frac{1}{\sqrt{2}}\partial X^+\theta_L\,,$$
$$q_L^{(1,2)} = -\frac{1}{12}(\theta_R\partial\theta_R)\theta_L + \frac{1}{24}\left\{(\theta_L\sigma^I\partial\theta_R) + (\theta_R\bar{\sigma}^I\partial\theta_L)\right\}\sigma^I\theta_R\,. \tag{C.2}$$

If $\mathcal{O}^{(n)}$ contains a term with charge $(n+1, -1)$ then the first term on the left hand side of eq. (C.1) will produce a term with charge $(n, -1)$.

$$\mathrm{Res}_{z\to w}q_L^{(-1,0)}(z)\mathcal{O}^{(n)}(w) \to \mathcal{L}^{(n,-1)}(w)\,, \tag{C.3}$$

which needs to be cancelled by similar contributions coming from the rest of the terms. There can not be any contribution coming from the last term. This is because the only operator of negative left (right)-charge is p_L (p_R) which has left-charge (right-charge) -1 and dimension 1 and $\mathcal{O}^{(n)}$ has dimension 1 for any n. Therefore,

$$\mathrm{Res}_{z\to w}\left[q_L^{(0,1)}(z)\mathcal{O}^{(n-2)}(w) + q^{(1,0)}(z)\mathcal{O}^{(n-2)}(w)\right] \to -\mathcal{L}^{(n,-1)}(w)\,. \tag{C.4}$$

In order for the first term to contribute to the right hand side $\mathcal{O}^{(n-2)}$ needs to have a term with charge $(n, -2)$, which is not possible for the same reason described above. Also the second term can not contribute, because $q_L^{(1,0)}$ does not have a residue with a dimension one term with charge $(n-1, -1)$. Therefore we must have,

$$\mathcal{L}^{(n,-1)}(z) = 0\,, \tag{C.5}$$

which implies $\mathcal{O}^{(n)}$ can not have a term with charge $(n+1, -1)$ for $n > -1$. The above argument is invalid for $n = -1$. This is because the right side of eq. (C.3) is trivial and the last two terms on the left side of eq. (C.1) do not exist. Therefore $\mathcal{O}^{(-1)}$ can have a term with charge $(0, -1)$ while satisfying eq. (C.1).

Let us now turn to prove the other part of the theorem, namely the term with charge $(n, 0)$ does not appear in $\mathcal{O}^{(n)}$ for $n > 1$. If $\mathcal{O}^{(n)}$ has a term with charge $(n, 0)$ then the first term in eq. (C.1) will produce a term with charge $(n-1, 0)$

$$\mathrm{Res}_{z\to w}q_L^{(-1,0)}(z)\mathcal{O}^{(n)}(w) \to \mathcal{K}^{(n-1,0)}(w)\,, \tag{C.6}$$

which needs to be cancelled by a similar term produced by the last two terms in eq. (C.1). As argued previously the last term in eq. (C.1) can not produce such a term. This implies,

$$\mathrm{Res}_{z\to w}\left[q_L^{(0,1)}(z)\mathcal{O}^{(n-2)}(w)+q_L^{(1,0)}(z)\mathcal{O}^{(n-2)}(w)\right]\to -\mathcal{K}^{(n-1,0)}(w)\ . \tag{C.7}$$

Let us first consider the first term on the left hand side of (C.7). In order for this term to contribute to the right hand side $\mathcal{O}^{(n-2)}$ should necessarily have a term with charge $(n-1,-1)$. Recalling that we are considering $n>1$, this requirement can not be satisfied due to the part of the theorem that has been proved first. We now consider the second term on the left hand side. From eqs. (C.2) it is easy to argue that in order for the second term to produce an $(n-1,0)$ term, it is necessary that $\mathcal{O}^{(n-2)}$ have an $(n-2,0)$ term with a factor of ∂X^-. Below we shall argue that the following statement is true:

The DDF vertices $\mathcal{B}^I(k^-,z)$ and $\mathcal{F}^a_L(k^-,z)$ do not contain the operator $\partial X^-(z)$ in their θ-expansion. (C.8)

Assuming this result for the time being we conclude that the second term on the left hand side of (C.7) does not contribute to the right hand side. This implies,

$$\mathcal{K}^{(n-1,0)}(z)=0\,, \tag{C.9}$$

which implies $\mathcal{O}^{(n)}(z)$ does not have a term with charge $(n,0)$ for $n>1$. For $n=0,-1$ the first term in eq. (C.1) gives zero for $(0,0)$ and $(-1,0)$ terms coming from $\mathcal{O}^{(0)}$ and $\mathcal{O}^{(-1)}$ respectively and the rest of terms are nonexistent in both the cases. Therefore $\mathcal{O}^{(0)}$ and $\mathcal{O}^{(-1)}$ can have terms with charges $(0,0)$ and $(-1,0)$ respectively while satisfying eq. (C.1).

We now proceed to prove the result (C.8). Let us first consider $\mathcal{B}^I(k^-,z)$. Any term which will give rise to $\partial X^-(z)$ in $\mathcal{B}^I(k^-,z)$ should come from a covariant term of the following form in $\mathcal{B}^\mu(k,z)$: $\partial X_\nu(z)\mathcal{A}^{\mu\nu}(k,z)$, where $\mathcal{A}^{\mu\nu}(k,z)$ is a dimension zero operator in the fermionic matter sector, and therefore constructed entirely out of θ's. The simplest possibility is $k^\mu k^\nu$ which does not have any θ. For the momentum restriction relevant for $\mathcal{B}^I(k^-,z)$, this gives rise to ∂X^+, not ∂X^-. To look for terms with non-zero number of θ's we should keep in mind that we must have even number of θ's in a given term and that, because the gamma matrices are symmetric, only a third rank tensor $(\theta\bar{\gamma}^{\mu\nu\rho}\theta)$ (which will be called θ^2 hereafter) can be constructed out of two θ's. Therefore an eligible term will be a product of such third rank tensors and momenta. The two vector indices in $\mathcal{A}^{\mu\nu}(k,z)$ can, in general, come from any such factors. It is easy to see that if any of them comes from momentum then the term either does not contribute to $\mathcal{B}^I(k^-,z)$ at all or gives rise to ∂X^+, not ∂X^-. Also the momentum independent terms can be ignored as we know from eq. (B.9) that at zero momentum there is no ∂X^- in $\mathcal{B}^I(k^-,z)$. The other possibilities include two cases where both the vector indices come from the same θ^2 factor and two different θ^2 factors. In the first case we have one vector index from the relevant θ^2 factor which is contracted with another θ^2 factor or monemtum. In the second case each of the two relevant θ^2 factors will have two vector indices contracted with other θ^2 factors and/or momenta. In order to have a ∂X^- in $\mathcal{B}^I(k^-,z)$ one of the vector index has to be $+$. Therefore we have a situation where we need to have a θ^2 factor with one vector index to be $+$ and one or two (depending on the cases described above) other vector indices to be contracted with other θ^2 factors and/or momenta. It is easy get convinced that a full contraction of these kinds will always involve momentum contraction(s). Since the only nonzero component of momentum is k^- these momentum contractions will always induce a $+$ index in the original θ^2 factor which had a free $+$ index. Since the θ^2 factor is antisymmetric in its indices this must be zero. This establishes that $\partial X^-(z)$ does not appear in $\mathcal{B}^I(k^-, z)$.

Let us now consider $\mathcal{F}^a_L(k^-, z)$. The covariant term in $\mathcal{F}_\alpha(k, z)$ that will potentially give rise to ∂X^- in $\mathcal{F}^a_L(k^-, z)$ should have the form: $\partial X_\mu \mathcal{D}^\mu_\alpha(k, z)$, where $\mathcal{D}^\mu_\alpha(k, z)$ is a dimensionless operator constructed entirely out of θ's. The possibilities are as follows:

$$\begin{array}{cc}
\text{Class I} & \text{Class II} \\
(\bar\gamma^\mu\theta)_\alpha\, \mathcal{E}_0(k, z) & (\bar\gamma_\rho\theta)_\alpha\, \mathcal{E}_1^{\mu\rho}(k, z) \\
(\bar\gamma^\mu{}_{\rho_1\rho_2}\theta)_\alpha\, \mathcal{E}_2^{[\rho_1\rho_2]}(k, z) & (\bar\gamma_{\rho_1\rho_2\rho_3}\theta)_\alpha\, \mathcal{E}_3^{\mu[\rho_1\rho_2\rho_3]}(k, z) \\
(\bar\gamma^\mu{}_{\rho_1\rho_2\rho_3\rho_4}\theta)_\alpha\, \mathcal{E}_4^{[\rho_1\rho_2\rho_3\rho_4]}(k, z) & (\bar\gamma_{\rho_1\rho_2\rho_3\rho_4\rho_5}\theta)_\alpha\, \mathcal{E}_5^{\mu[\rho_1\rho_2\rho_3\rho_4\rho_5]}(k, z)
\end{array}$$

where all the operators denoted by $\mathcal{E}$ with various tensor structures are products of θ^2 terms and momenta. None of the class I operators appears in $\mathcal{F}^a_L(k^-, z)$ when the free vector index is set to $+$. This is simply because the prefactor linear in θ that appears in each of these operators is projected to the "wrong" chirality. Although similar projection gives the "right" chirality for the class II operators, they do not appear because of the momentum restriction involved in the $\mathcal{E}$ operators. Each of the $\mathcal{E}$ operators involves a θ^2 factor whose one vector index is set to $+$ and one or two other indices are contracted to other θ^2 factors and/or momenta. We have argued before that such terms are zero. This establishes that $\partial X^-(z)$ does not appear in $\mathcal{F}^a_L(k^-, z)$.

D. The instantonic D-branes

Here we shall discuss the boundary conditions and boundary states for both the BPS and non-BPS instantonic D-branes in type IIB string theory. Following [11] we shall work in the free CFT.

In the BPS case, boundary condition for the bosonic matter part of the CFT is , as usual, given by (on UHP),

$$\partial X^\mu(z) = -(M^V)^\mu{}_\nu \bar\partial X^\nu(\bar z)\,, \qquad \text{at } z = \bar z\,, \tag{D.1}$$

where M^V is the diagonal reflection matrix with -1 for the Neumann directions and $+1$ for the Dirichlet directions. For the fermionic matter and bosonic ghost sectors the boundary conditions can be obtained by demanding that the scalars and vectors constructed out of the fields in these sectors are related at the boundary in the following way,

$$\Phi(z) = \tilde\Phi(\bar z)\,, \qquad A^\mu(z) = -(M^V)^\mu{}_\nu \tilde A^\nu(\bar z)\ . \tag{D.2}$$

The result is,

$$U^\alpha(z) = -i\eta (M^S)^\alpha{}_\beta \tilde{U}^\beta(\bar{z})\,, \qquad V_\alpha(z) = i\eta(\bar{M}^S)_\alpha{}^\beta \tilde{V}_\beta(\bar{z})\,, \qquad \text{at } z=\bar{z}\,, \tag{D.3}$$

where,

$$M^S = \gamma^{I_1 I_2 \cdots I_{p+1}}\,, \qquad \bar{M}^S = \bar{\gamma}^{I_1 I_2 \cdots I_{p+1}}\,, \tag{D.4}$$

with p being odd and $I_1, I_2, \ldots I_{p+1}$ being the Neumann directions (all spatial). $\eta = \pm 1$ correspond to brane and anti-brane. Since the matrices M^S and $\bar{M}^S$ includes only the spatial directions we have the following properties,

$$\begin{aligned} M^S (\bar{M}^S)^T &= \mathbb{I}_{16}\,, \\ M^S \gamma^\mu (M^S)^T &= (M^V)^\mu{}_\nu \gamma^\nu\,, \qquad (M^S)^T \bar{\gamma}^\mu M^S = (M^V)^\mu{}_\nu \bar{\gamma}^\nu\,, \\ \bar{M}^S \bar{\gamma}^\mu (\bar{M}^S)^T &= (M^V)^\mu{}_\nu \bar{\gamma}^\nu\,, \qquad (\bar{M}^S)^T \gamma^\mu \bar{M}^S = (M^V)^\mu{}_\nu \gamma^\nu\ . \end{aligned} \tag{D.5}$$

All the above equations differ by a sign with respect to the case where the matrices include the time direction as in [11]. The BRST and supersymmetry currents are related on the boundary in the following way,

$$j_B(z) = \tilde{j}_B(\bar{z})\,, \qquad q_\alpha(z) = i\eta(\bar{M}^S)_\alpha{}^\beta \tilde{q}_\beta(\bar{z})\ . \tag{D.6}$$

As a result the BPS boundary state $|\mathrm{Inst}_p, \eta\rangle_{\mathrm{BPS}}$ is BRST invariant and preserves the expected combination of the supersymmetry,

$$\left(Q_B + \tilde{Q}_B\right) |\mathrm{Inst}_p, \eta\rangle_{\mathrm{BPS}} = 0\,, \qquad \left(Q_\alpha + i\eta(\bar{M}^S)_\alpha{}^\beta \tilde{Q}_\beta\right) |\mathrm{Inst}_p, \eta\rangle_{\mathrm{BPS}} = 0\ . \tag{D.7}$$

As we have seen in [15], unlike the case of BPS D-branes, open string boundary conditions for non-BPS D-branes do not involve the spinor matrices representing reflections along Neumann directions. Therefore these boundary conditions, once written in terms of the vector matrix M^V, should look the same for both Lorentzian and instantonic D-branes. Indeed the bosonic matter and combined fermionic matter and bosonic ghost parts of the non-BPS D-instanton boundary conditions are given by eqs. (D.1), (2.1) respectively with M^V representing reflections along the Neumann directions of the considered non-BPS D-brane.

The boundary states for both the BPS and non-BPS instantonic D-branes can be constructed explicitly in terms of the oscillators, as was done in [11], but we do not need the explicit expression for the purpose of the present paper. All we need is to argue using the boundary conditions (D.3) and equations (D.5) that any holomorphic spinor with either upper or lower spinor index will be related to the corresponding anti-holomorphic one following the same rule as followed in eqs. (D.3).

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